# MATH 512, FALL 14 COMBINATORIAL SET THEORY WEEK 10

Let $\kappa \leq \lambda$ be regular cardinals. $\mathcal{P}_\kappa(\lambda)$ is the set of all $x \subset \lambda$ with $|x| < \kappa$. A set $C \subset \mathcal{P}_\kappa(\lambda)$ is club if

- unboundedness: for every $x \in \mathcal{P}_\kappa(\lambda)$, there is $y \in C$ with $x \subset y$,
- closed: if $\tau < \kappa$ and $\langle x_i \mid i < \tau \rangle$ are elements in $C$ such that $i < k$ implies $x_i \subset x_k$, then $\bigcup_{i<\tau} x_i \in C$.

A set is stationary in $\mathcal{P}_\kappa(\lambda)$ if it meets every club.

Given $\langle A_\alpha \mid \alpha < \lambda \rangle$ with every $A_\alpha \subset \mathcal{P}_\kappa(\lambda)$, their *diagonal intersection* is $\triangle_{\alpha<\lambda} A_\alpha := \{x \in \mathcal{P}_\kappa(\lambda) \mid x \in \bigcap_{\alpha \in x} A_\alpha\}$.

The following two lemmas are generalizations of the case for club filters on $\kappa$. The proof is left as an exercise (you can also see Chapter 8 of Jech).

**Lemma 1.** *The club filter on $\mathcal{P}_\kappa(\lambda)$ is $\kappa$-complete and closed under diagonal intersections.*

**Lemma 2.** *(Fodor) Suppose that $S$ is a stationary subset of $\mathcal{P}_\kappa(\lambda)$ and $f : S \to \lambda$ is regressive i.e for every $x \in S$, $f(x) \in x$. Then there is a stationary $T \subset S$, such that $f$ is constant on $T$.*

**Definition 3.** *$U \subset \mathcal{P}(\mathcal{P}_\kappa(\lambda))$ is called a normal measure on $\mathcal{P}_\kappa(\lambda)$ if:*

1. *$U$ is a nonprincipal ultrafilter;*
2. *$U$ is $\kappa$-complete i.e. whenever $\langle A_i \mid i < \tau \rangle$, $\tau < \kappa$ are sets in $U$, then so is $\bigcap_{i<\tau} A_i$;*
3. *normality: whenever $\langle A_\alpha \mid \alpha < \lambda \rangle$, are sets in $U$, then so is their diagonal intersection, $\triangle_{\alpha<\lambda} A_\alpha$.*

An equivalent condition for normality is that for every $S \in U$ and regressive function $f : S \to \lambda$, there is $A \subset S$, $A \in U$, such that $f$ is constant on $A$.

**Definition 4.** *A cardinal $\kappa$ is $\lambda$-supercompact if there is a normal measure on $\mathcal{P}_\kappa(\lambda)$; $\kappa$ is supercompact if it is $\lambda$-supercompact for every $\lambda$.*

**Theorem 5.** *A cardinal $\kappa$ is $\lambda$-supercompact iff there is an elementary embedding $j : V \to M$ with critical point $\kappa$ such that $j(\kappa) > \lambda$, $M^\lambda \subset M$.*

*Proof.* For the first direction, suppose that $U$ is a normal measure on $\mathcal{P}_\kappa(\lambda)$. Let $j : V \to Ult(V, U)$ be given by $j(a) = [c_a]_U$. Here $c_a$ is the constant function on $\mathcal{P}_\kappa(\lambda)$ with value $a$. Then, $\phi(a_1, ..., a_n)$ holds in $V$ iff $\{x \in \mathcal{P}_\kappa(\lambda) \mid \phi(c_{a_1}(x), ..., c_{a_n}(x))\} = \mathcal{P}_\kappa(\lambda) \in U$ iff (by Los), $M \models \phi(j(a_1), ...j(a_n))$. So, $j$ is elementary. Note that since $U$ is $\kappa$-complete, the ultrapower $Ult(V, U)$ is wellfounded, so we can identify it with its transitive collapse.

**Claim 6.** *For all $\alpha < \kappa$, $j(\alpha) = \alpha$.*

*Proof.* By induction on $\alpha < \kappa$. Suppose that $\beta < \kappa$, and for all $\alpha < \beta$, $j(\alpha) = [c_\alpha]_U = \alpha$. If $\beta = \alpha + 1$, by elementarity, $j(\beta) = j(\alpha) + 1 = \alpha + 1$. Suppose that $\beta$ is limit. First, for every $\alpha < \beta$, $\alpha = [c_\alpha] <_U [c_\beta]_U$, so $\beta \leq [c_\beta]_U$. Also, if $[f]_U < [c_\beta]_U$, then $\{x \mid f(x) < \beta\} \in U$. Since $U$ is $\kappa$-complete, for some $\alpha < \beta$, $\{x \mid f(x) = \alpha\} \in U$, i.e. $[f]_U = [c_\alpha]_U$. So $[c_\beta]_U = \sup_{\alpha<\beta}[c_\alpha] = \sup_{\alpha<\beta} \alpha = \beta$.

□

Note that $X := \{x \mid \kappa \cap x \in \kappa\}$ is a measure one set; for $x \in X$, write $\kappa_x := \kappa \cap x$.

**Claim 7.** $\kappa = [x \mapsto \kappa_x]_U < j(\kappa)$.

*Proof.* If $\gamma < \kappa$, then by the above claim $\gamma = [c_\gamma]$. Since $\{x \mid \gamma < \kappa_x\} \in U$, we have that $\gamma < [x \mapsto \kappa_x]_U$. I.e. $\kappa \leq [x \mapsto \kappa_x]_U$. On the other hand if $[g]_U < [x \mapsto \kappa_x]_U$, then for almost all $x$, $g(x) < \kappa_x$. By normality, there is some $\gamma < \kappa$, such that for almost all $x$, $g(x) = \gamma$. So, $[x \mapsto \kappa_x]_U \leq \kappa$.

□

It follows that the critical point of $j$ is $\kappa$.

**Claim 8.** $j"\lambda = [x \mapsto x]_U$.

*Proof.* Let $\gamma < \lambda$. Since $\{x \mid \gamma \in x\} \in U$, we have that $j(\gamma) \in [x \mapsto x]_U$; i.e. $j"\lambda \subseteq [x \mapsto x]_U$.

For the other direction, let $[g] \in [x \mapsto x]_U$. Then for almost all $x$, $g(x) \in x$, so by normality, there is some $A \in U$, such that $g$ is constant on $A$. Say $\gamma < \lambda$ is such that $g(x) = \gamma$ for all $x \in A$. That means that $[g] = j(\gamma)$; i.e. $[g] \in j"\lambda$.

□

**Claim 9.** $\lambda = [x \mapsto o.t.(x)]_U < j(\kappa)$.

*Proof.* Let $[g]_U = \lambda$. Since $M \models [g] = \lambda = o.t.(j"\lambda) = o.t.([x \mapsto x]_U)$, by Los, we have that for almost all $x$, $g(x) = o.t.(x)$. So, $\lambda = [x \mapsto o.t.(x)]_U < j(\kappa)$. □

By a similar argument, for $\kappa < \tau < \lambda$, we have $\tau = [x \mapsto o.t.(x \cap \tau)]$; $\sup(j"\lambda) = [x \mapsto \sup(x)]_U$.

The last thing to show is that $M^\lambda \subset M$. For that, suppose that $\langle [f_i] \mid i < \lambda \rangle$ is a sequence of elements in $M$. Let $f : \mathcal{P}_\kappa(\lambda) \to V$ be

$$f(x) = \langle f_i(x) \mid i \in x \rangle.$$

Then $[f] \in M$. Also since every $f(x)$ is a sequence of length $o.t.(x)$, by Los's theorem, in $M$, $[f]$ is a sequence of length $\lambda$. For every $x$, let $i_x := o.t.(x \cap i)$. By the above remark, for every $i < \lambda$, $[x \mapsto i_x]_U = i$.

In $V$, $\{x \mid \text{ the } i_x - \text{th element of } f(x) = f_i(x)\} = \{x \mid i \in x\} \in U$. So, by Los, the $i$-th element of the sequence $[f]$ is exactly $[f_i]$. It follows that $[f] = \langle [f_i] \mid i < \kappa \rangle \in M$.

For the other direction of the theorem, fix an embedding $j : V \to M$ as in the assumption. Let

$$U := \{X \subset \mathcal{P}_\kappa(\lambda) \mid j"\lambda \in j(X)\}.$$

It is straightforward by elementarity to check that $U$ is an ultrafilter. For $\kappa$-completeness: suppose that $\tau < \kappa$ and $\langle X_\alpha \mid \alpha < \tau\rangle$ are sets in $U$. Then for every $\alpha$, $j"\lambda \in j(X_\alpha)$. Since $j(\tau) = \tau$, we have that $j(\langle X_\alpha \mid \alpha < \tau\rangle) = \langle j(X_\alpha) \mid \alpha < \tau\rangle$. Then $j"\lambda \in \bigcap_{\alpha<\tau} j(X_\alpha) = j(\bigcap_{\alpha<\tau} X_\alpha)$, and so $\bigcap_{\alpha<\tau} X_\alpha \in U$.

For normality, suppose that $f : \mathcal{P}_\kappa(\lambda) \to \lambda$ is a regressive function. Then $X = \{x \mid f(x) \in x\} \in U$, and so $j"\lambda \in j(X)$. It follows that $jf(j"\lambda) \in j"\lambda$, so for some $\gamma < \lambda$, $jf(j"\lambda) = j(\gamma)$. It follows that $Y := \{x \mid f(x) = \gamma\} \in U$, since $j"\lambda \in j(Y)$.

□

**Lemma 10.** *Let $\kappa$ be $\lambda$-supercompact and $U$ be a normal measure on $\mathcal{P}_\kappa(\lambda)$. Let $j : V \to M$ be the ultrapower embedding $j = j_U$. Then for every $f : \mathcal{P}_\kappa(\lambda) \to V$, $[f]_U = jf(j"\lambda)$.*

*Proof.* As shown above, $[x \mapsto x]_U = j"\lambda$. Now for any $f : \mathcal{P}_\kappa(\lambda) \to V$, $[g] \in [f]$ iff $\{x \mid g(x) \in f(x)\} \in U$ iff by Los, $[g] \in [x \mapsto f]([x \mapsto x]) = jf(j"\lambda)$.

Also note that for any $X \subset \mathcal{P}_\kappa(\lambda)$, we have that $X \in U$ iff $[x \mapsto x]_U \in j(X)$ iff $j"\lambda \in j(X)$.

□

Let $j : V \to M$ be an elementary embedding with critical point $\kappa$, $j(\kappa) > \lambda$, $M^\lambda \subset M$. Define $U := \{X \subset \mathcal{P}_\kappa(\lambda) \mid j"\lambda \in j(X)\}$ and $k : Ult(U, V) \to M$ by

$$k([f]_U) = jf(j"\lambda).$$

Then $k$ is elementary and $j = k \circ j_U$. (Here $j_U(a) = [c_a]_U$.)